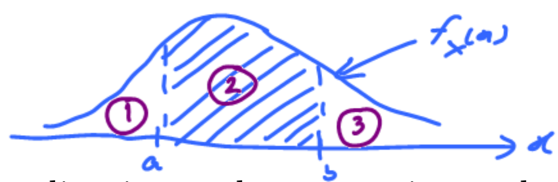
10.2 Properties of PDF and CDF for Continuous Random Variables

10.18. $p_X(x) = P[X = x] = P[x \le X \le x] = \int_x^x f_X(t)dt = 0$.

Again, it makes no sense to speak of the probability that X will take on a pre-specified value. This probability is always zero.

10.19. $P[X = a] = P[X = b] = 0$. Hence,

$$F_X(b) - F_X(a) = P[a < X < b] = P[a \le X < b] = P[a < X \le b] = P[a \le X \le b]$$



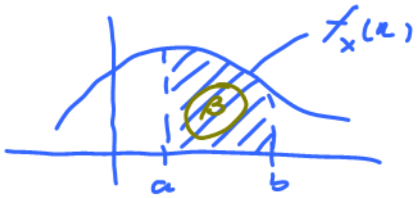
$F_X(b) = \int_{-\infty}^{b} f_X(x)dx = ① + ②$

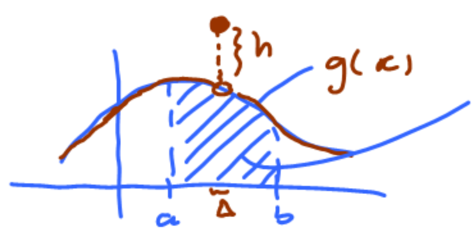
$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = ①$

- The corresponding integrals over an interval are not affected by whether or not the endpoints are included or excluded. [41]

10.20. The pdf f_X is determined only almost everywhere[42]. Given a pdf f for a continuous random variable X, if we construct a function g by changing the function f at a countable number of points[43], then g can also serve as a pdf for X.

This is because f_X is defined via its integration property. Changing the value of a function at a few points does not change its area under the curve (from a to b)





$\text{Area} = \beta + \overbrace{(h \times \Delta)}^{0}$

$= \beta$

10.21. The cdf of any kind of random variable X is defined as

$$F_X(x) = P[X \le x].$$

Note that even though there are more than one valid pdfs for any given random variable, the cdf is unique. There is only one cdf for each random variable.

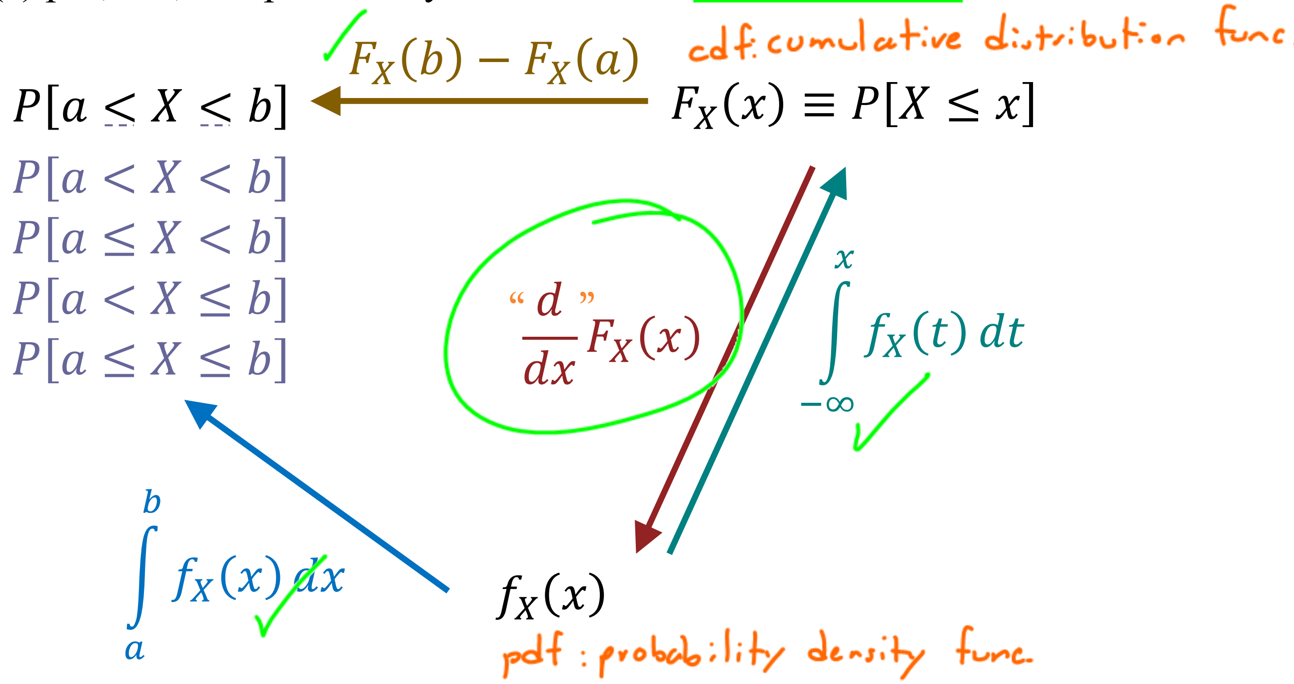
---

[41] This implies that, when we work with continuous random variables, it is usually not necessary to be precise about specifying whether or not a range of numbers includes the endpoints. This is quite different from the situation we encounter with discrete random variables where it is critical to carefully examine the type of inequality.

[42] Lebesgue-a.e, to be exact

[43] More specifically, if $g = f$ Lebesgue-a.e., then g is also a pdf for X.

(a) pdf, cdf, and probability calculation for continuous RV



(b) Finding Probabilities from CDF

Definition: $F_X(x) \equiv P[X \leq x]$

For **any RV**,	For **continuous RV**,
• $P[X \leq b] = F_X(b)$ $P[X < b] = F_X(b) - P[X = b]$	• $P[X \leq b] = F_X(b)$ $P[X < b] = F_X(b)$
• $P[X > a] = 1 - F_X(a)$ $P[X \geq a] = 1 - F_X(a) + P[X = a]$	• $P[X > a] = 1 - F_X(a)$ $P[X \geq a] = 1 - F_X(a)$
• $P[a < X \leq b] = F_X(b) - F_X(a)$	• $P[a < X \leq b] = F_X(b) - F_X(a)$ $P[a < X < b] = F_X(b) - F_X(a)$ $P[a \leq X < b] = F_X(b) - F_X(a)$ $P[a \leq X \leq b] = F_X(b) - F_X(a)$
• $P[X = a] = F_X(a) - F_X(a^-)$ (amount of jump in the CDF @ a)	• $P[X = a] = 0$

Figure 29: Summary of properties involving CDF

10.22. Unlike the cdf of a discrete random variable, the cdf of a continuous random variable has no jump and is continuous everywhere.

10.23. For continuous random variable, given the pdf $f_X(x)$, we can find the cdf of X by

pdf $\underset{\frac{d}{dx}}{\overset{\int}{\rightleftarrows}}$ cdf

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)dt.$$

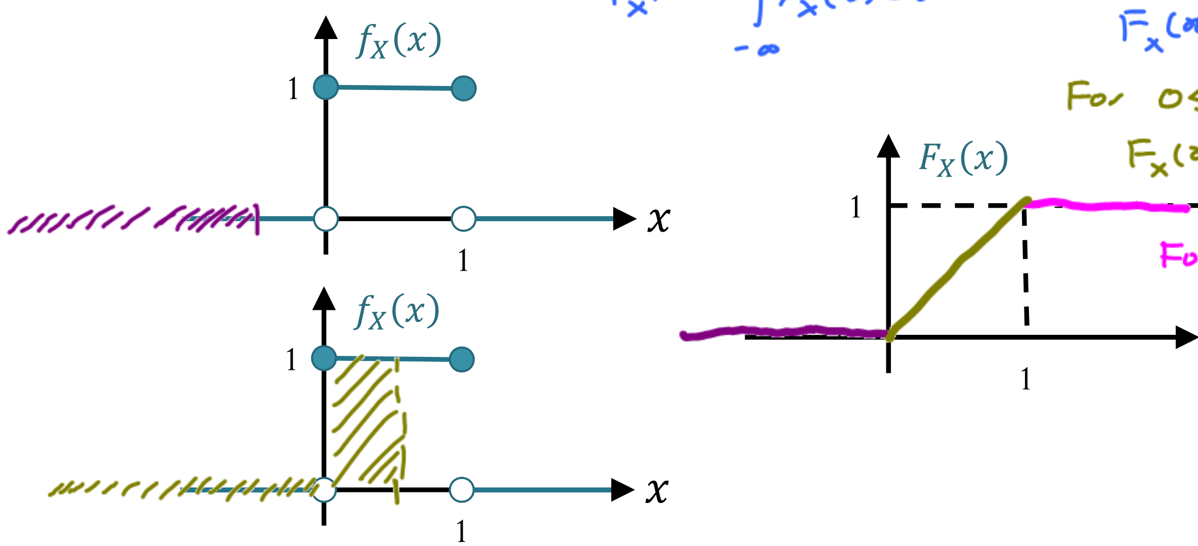
Example 10.24. For the random variable generated by the `rand` command in `MATLAB` or the `rand()` command in `Excel`,

$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$

For $x < 0$, $F_X(x) = 0$.

For $0 \le x \le 1$, $F_X(x) = x$

For $x > 1$, $F_X(x) = 1$



10.25. Given the cdf $F_X(x)$ of a continuous random variable, we can find the pdf $f_X(x)$ by

Step 1 If F_X is differentiable at x, we set

$$\frac{d}{dx}F_X(x) = f_X(x).$$

Step 2 From 10.20, at countably many points, we can set the values of f_X to be any value. We use this to deal with the boundary

Ex. $$f_X(x) = \begin{cases} \frac{3}{26}x^2, & x\in(1,3), \\ 0, & \text{otherwise.}\end{cases}$$

pdf

$$F_X(x) = ? \qquad \equiv P[X\le x] = \int_{-\infty}^{x} f_X(t)\,dt$$

① $x<1$

$$F_X(x) = \int_{-\infty}^{x} \underbrace{f_X(t)}_{0}\,dt = 0$$

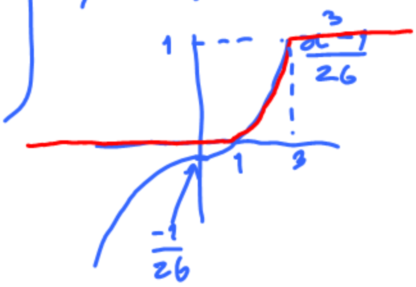
② $1 < x < 3$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\cancel{\int_{-\infty}^{1} \underbrace{f_X(t)}_{0}\,dt}}_{0} + \int_{1}^{x} \underbrace{f_X(t)}_{\frac{3}{26}t^2}\,dt = \frac{\cancel{3}}{26}\,\frac{t^3}{\cancel{3}}\Bigg|_{1}^{x} = \frac{x^3-1}{26}$$

③ $x > 3$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\cancel{\int_{-\infty}^{1}}}_{0} + \underbrace{\int_{1}^{3}}_{\frac{3^3-1}{26}=1} + \underbrace{\cancel{\int_{3}^{x} f_X(t)\,dt}}_{0} = 1$$

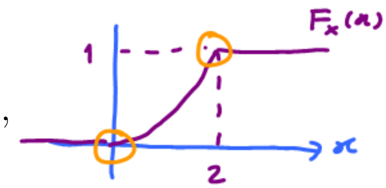
$$F_X(x) = \begin{cases} 0, & x<1, \\ \frac{x^3-1}{26}, & 1\le x\le 3, \\ 1, & x>3. \end{cases}$$



point(s) including the point(s) where F_X is not differentiable. Usually, the values are selected to give simple expression. (In many cases, they are simply set to 0.)

Example 10.26. Suppose that the lifetime X of a device has the cdf

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}x^2, & 0 \le x \le 2, \\ 1, & x > 2. \end{cases}$$



Because $F_x(x)$ is defined piecewise and the expression defining each piece is "nice", we can find the derivative for each piece and get

$\frac{d}{dx}$

$$f_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{2}, & 0 < x < 2, \\ 0, & x > 2. \end{cases}$$

This leaves two points[44] to be considered: $x = 0$ and $x = 2$. However, they are only two points and therefore, from 10.20, the values of the pdf can be any real numbers. Here, we set the values to be 0 at both points:

$$f_X(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

10.27. In many situations when you are asked to find the pdf from a description of a random variable, it may be easier to find cdf first and then differentiate it to get pdf.

Exercise 10.28. A point is "picked at random" in the inside of a circular disk with radius r. Let the random variable X denote the distance from the center of the disk to this point. Find $f_X(x)$.

[44] At each of these boundary points, the expressions on both of its sides are different and hence, to really find its derivative, we need to consider whether the derivative from the left exists and is the same as the derivative from the right. At $x = 0$, turn out that the slope on both sides is 0. So the derivative exists. However, at $x = 2$, F_X has no derivative: the slope is 1 from the left but 0 from the right.

10.29. f_X is nonnegative and $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Example 10.30. Random variable X has pdf

$$f_X(x) = \begin{cases} ce^{-2x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the constant c and sketch the pdf.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \cancel{\int_{-\infty}^{0} \underbrace{f_X(x)}_{0}\,dx} + \int_{0}^{\infty} \underbrace{f_X(x)}_{ce^{-2x}}\,dx = c\,\frac{e^{-2x}}{-2}\Big|_0^{\infty} = \frac{c}{-2}(0-1)$$

$$1 = \frac{c}{2}$$

$$c = 2$$

Example 10.31. The pdf of a random variable X is shown in Figure 30. What should be the value of h?

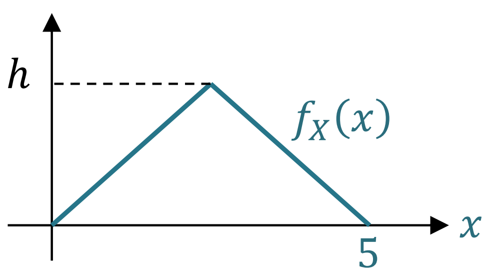


Figure 30: Triangular pdf for Example 10.31.

$$\text{area} = \frac{1}{2} \times 5 \times h = 1$$

$$h = \frac{2}{5}$$

Definition 10.32. A continuous random variable is called ***exponential*** if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \le 0 \end{cases}$$

for some $\lambda > 0$

Example 10.33. In Example 10.30, X is an exponential random variable with $\lambda = 2$.

Theorem 10.34. Any nonnegative[45] function that integrates to one is a ***probability density function*** (pdf) of some random variable [9, p.139].

[45]or nonnegative a.e.

10.3 Expectation and Variance

10.35. ***Expectation***: Suppose X is a continuous random variable with probability density function $f_X(x)$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) dx \tag{23}$$

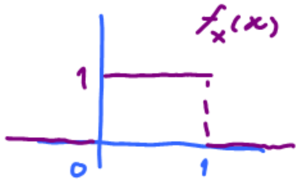
$$\mathbb{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{24}$$

In particular,

$$\mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

$$\operatorname{Var} X = \int_{-\infty}^{\infty} (x - \mathbb{E}X)^2 f_X(x) dx = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2.$$

Example 10.36. For the random variable generated by the `rand` command in `MATLAB` or the `rand()` command in `Excel`,



$$\mathbb{E}[X] \equiv \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x(1) dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2(1) dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$\operatorname{Var} X \equiv \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$$

Example 10.37. For the exponential random variable introduced in Definition 10.32,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbb{E}X \equiv \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx = \lambda\left(-\frac{x}{\lambda} e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x}\right)\Big|_0^{\infty}$$

Integration by parts:

$$\begin{array}{c|ccc|c} \frac{d}{dx} & x & \xrightarrow{+} & e^{-\lambda x} & \int \\ \downarrow & 1 & \xrightarrow{-} & \frac{1}{-\lambda} e^{-\lambda x} & \downarrow \\ & 0 & \xrightarrow{+} & \frac{1}{(-\lambda)^2} e^{-\lambda x} & \end{array}$$

$$= \lambda\Big(\underbrace{(0 - 0)}_{x=\infty} - \underbrace{\left(0 - \frac{1}{\lambda^2}\right)}_{x=0} \Big)$$

$$= \frac{1}{\lambda}$$

$$\lim_{x\to\infty} x e^{-\lambda x} = \lim_{x\to\infty} \frac{x}{e^{\lambda x}} \underset{\text{L'Hôpital's}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{\lambda x}} = \lim_{x\to\infty} \frac{1}{\lambda e^{\lambda x}} = 0$$

10.38. If we compare other characteristics of discrete and continuous random variables, we find that with discrete random variables, many facts are expressed as sums. With continuous random variables, the corresponding facts are expressed as integrals.

10.39. All of the properties for the expectation and variance of discrete random variables also work for continuous random variables as well:

(a) Intuition/interpretation of the expected value: As $n \to \infty$, the average of n independent samples of X will approach $\mathbb{E}X$. This observation is known as the "Law of Large Numbers".

(b) For $c \in \mathbb{R}$, $\mathbb{E}[c] = c$

(c) For constants a, b, we have $\mathbb{E}[aX + b] = a\mathbb{E}X + b$.

(d) $\mathbb{E}\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i \mathbb{E}[g_i(X)]$.

(e) $\operatorname{Var} X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2$

(f) $\operatorname{Var} X \geq 0$.

(g) $\operatorname{Var} X \leq \mathbb{E}\left[X^2\right]$.

(h) $\operatorname{Var}[aX + b] = a^2 \operatorname{Var} X$.

(i) $\sigma_{aX+b} = |a| \, \sigma_X$.

10.40. ***Chebyshev's Inequality***:

$$P\left[|X - \mathbb{E}X| \geq \alpha\right] \leq \frac{\sigma_X^2}{\alpha^2}$$

or equivalently

$$P\left[|X - \mathbb{E}X| \geq n\sigma_X\right] \leq \frac{1}{n^2}$$

- This inequality use variance to bound the "tail probability" of a random variable.
- Useful only when $\alpha > \sigma_X$

Example 10.41. A circuit is designed to handle a current of 20 mA plus or minus a deviation of less than 5 mA. If the applied current has mean 20 mA and variance 4 $(\text{mA})^2$, use the Chebyshev inequality to bound the probability that the applied current violates the design parameters.

Let X denote the applied current. Then X is within the design parameters if and only if $|X - 20| < 5$. To bound the probability that this does not happen, write

$$P\left[|X - 20| \geq 5\right] \leq \frac{\operatorname{Var} X}{5^2} = \frac{4}{25} = 0.16.$$

Hence, the probability of violating the design parameters is at most 16%.

10.42. Interesting applications of expectation:

(a) $f_X(x) = \mathbb{E}\left[\delta\left(X - x\right)\right]$

(b) $P\left[X \in B\right] = \mathbb{E}\left[1_B(X)\right]$

Example 10.43. Consider two distributions for a random variable X. In part (a), which corresponds to the second column in the table below, X is a *discrete* random variable with its pmf specified in the first row. In part (b), which corresponds to the third column, X is a *continuous* random variable with its pdf specified in the first row.

Distribution	$p_X(x) = \begin{cases} cx^2, & x \in \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$	$f_X(x) = \begin{cases} cx^2, & x \in (1,2), \\ 0, & \text{otherwise.} \end{cases}$
(i) Find c		
(ii) Find $\mathbb{E}X$		
(iii) Find $\mathbb{E}\left[X^2\right]$		
(iv) Find $\operatorname{Var} X$		
(v) Find σ_X		